Space charge calculations

Apr 15 2016

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Outlook:

- R-Distortions ALICE/STAR/PHENIX
- Charge density from FP

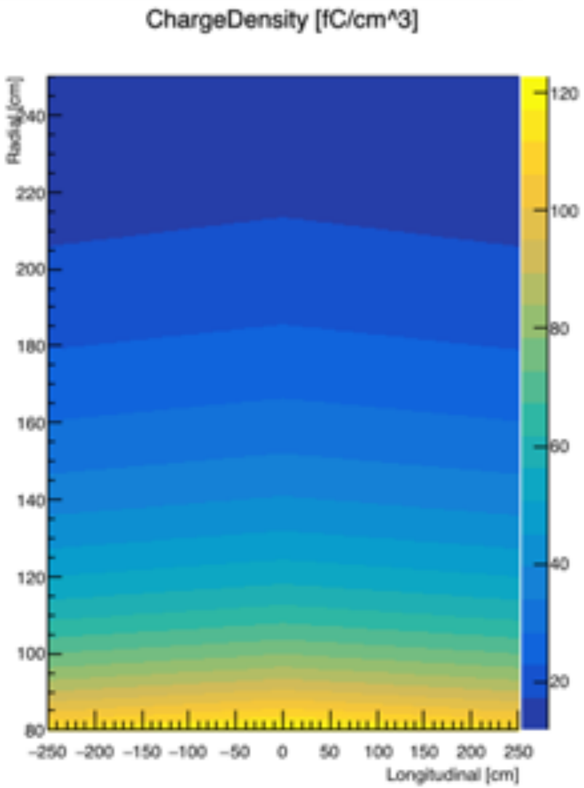
Methodology for distortions

- Initial Charge Density from toy model (ala ALICE)
- Er and Ez from Laplace solution to ICD in cage at V=0
- DeltaR from Langevin formalism using Ez = 400V/cm

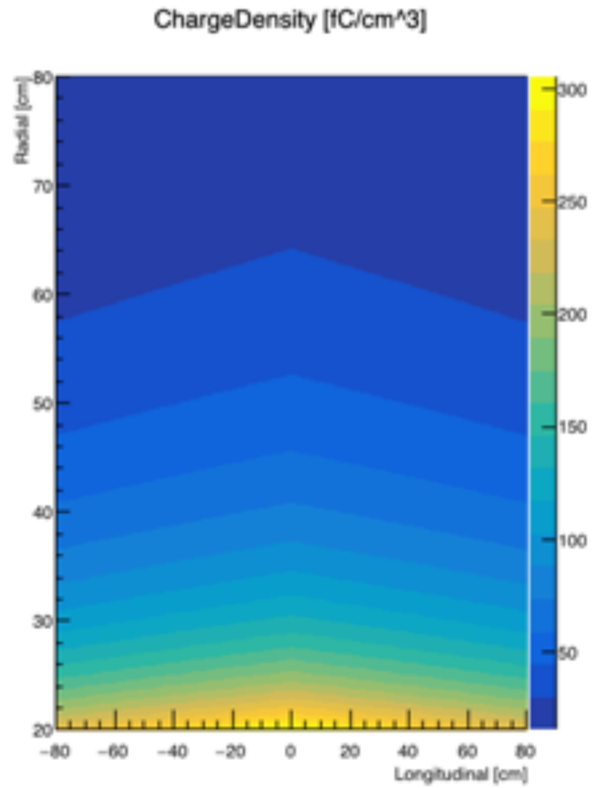
Thanks to Sourav, now the computation of E is done in parallel. This is a huge improvement!

Initial Charge Density

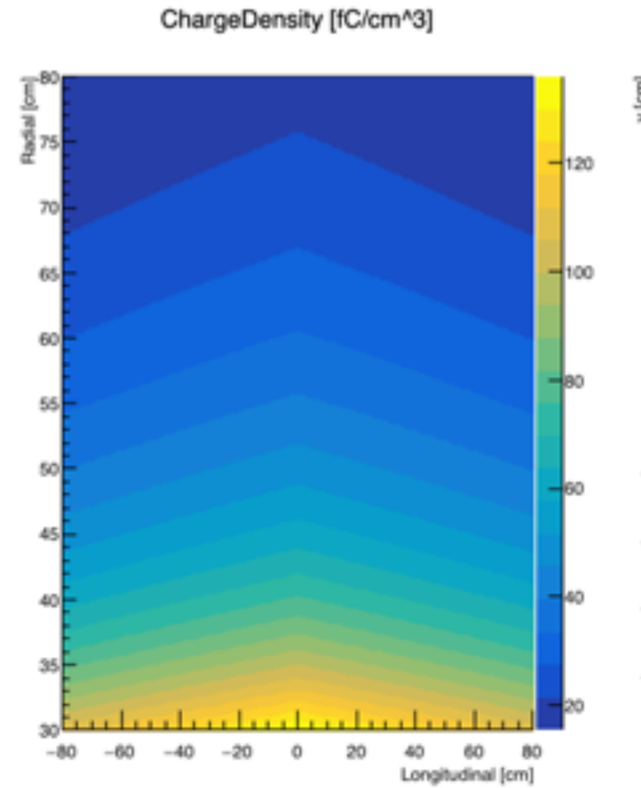
ALICE
Radial dependence set at 2
Gas factor at 1.0/76628.0
Multiplicity at 900
DC Rate at 50kHz
BackFlow at 20 (=1.0%2000)



sPHENIX20
Radial dependence set at 2
Gas factor at 1.0/76628.0
Multiplicity at 450
DC Rate at 50kHz
BackFlow at 6 (=0.3%2000)



sPHENIX30
Radial dependence set at 2
Gas factor at 1.0/76628.0
Multiplicity at 450
DC Rate at 50kHz
BackFlow at 6 (=0.3%2000)

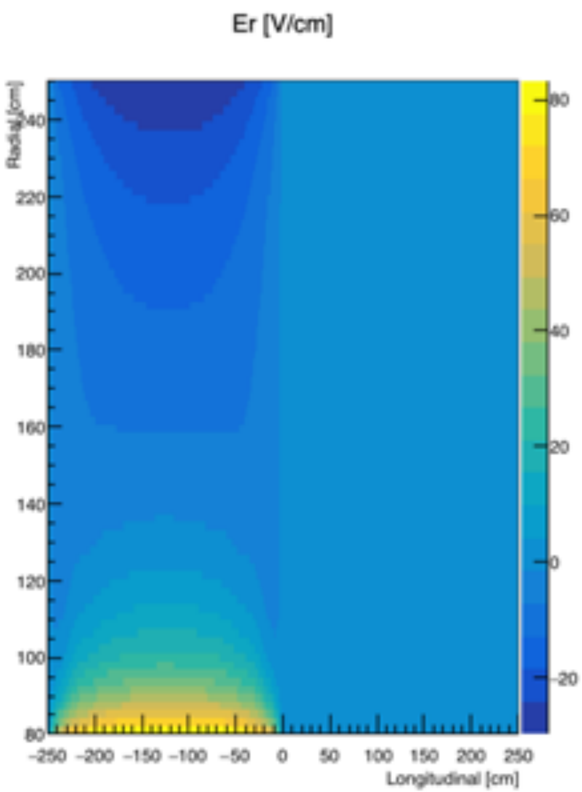


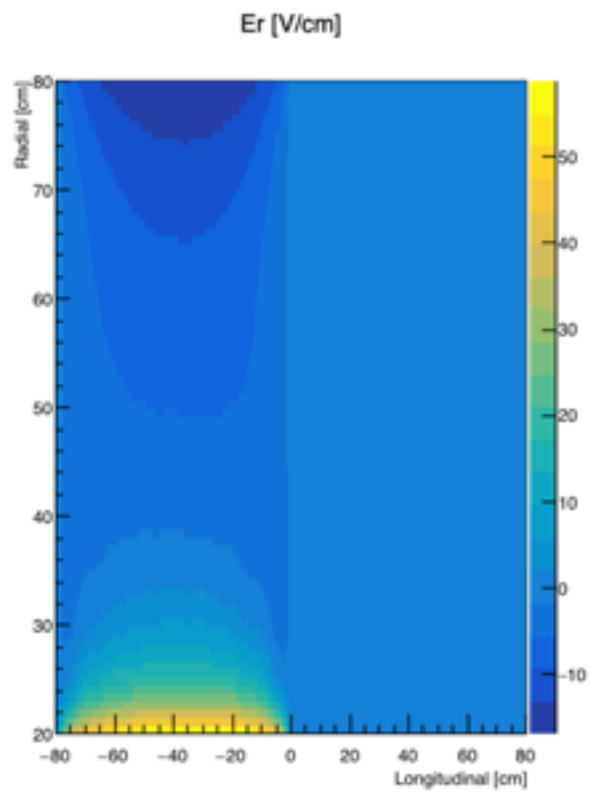
Induced Electric Field

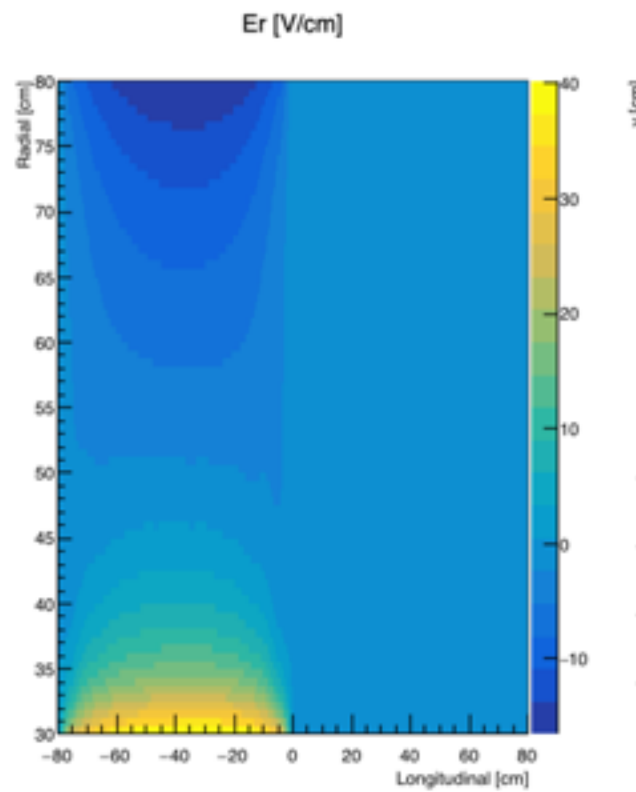
ALICE
Grid size:
Rad = 2.13 cm
Phi = 360 deg
Lon = 2 cm

sPHENIX20
Grid size:
Rad = 0.75 cm
Phi = 360 deg
Lon = 0.64 cm

sPHENIX30
Grid size:
Rad = 0.63 cm
Phi = 360 deg
Lon = 0.64 cm

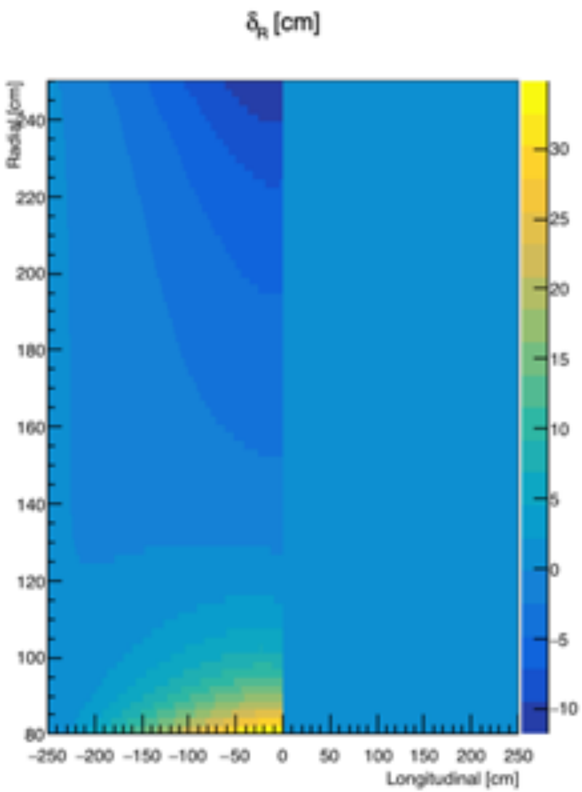




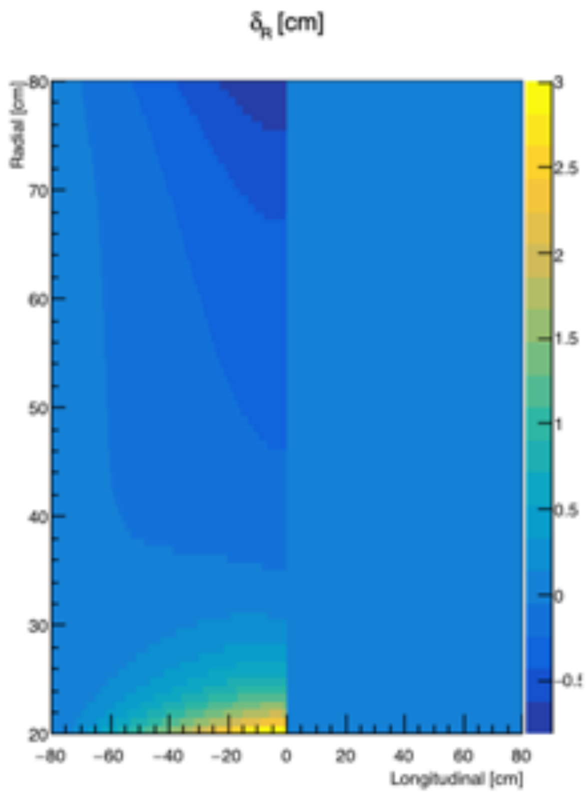


Estimated mean distortions in R

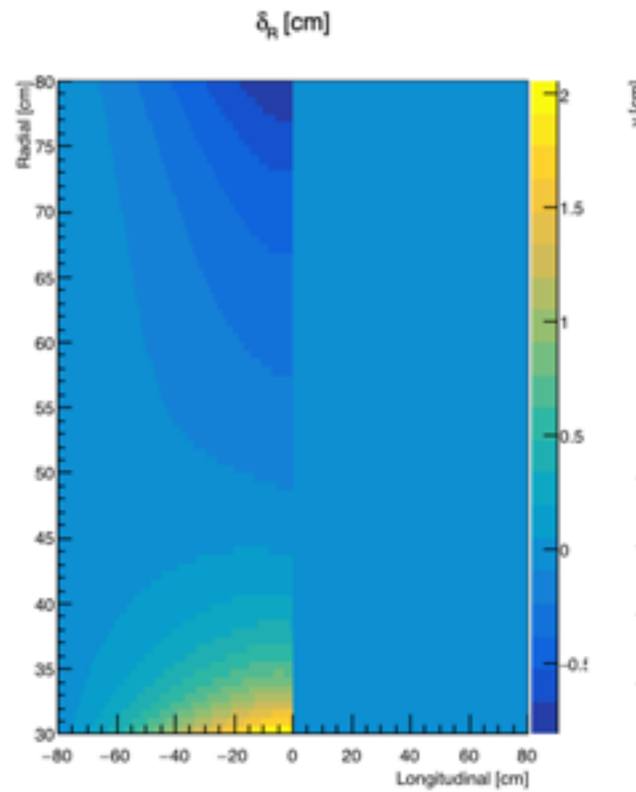
ALICE
Grid size:
Rad = 2.13 cm
Phi = 360 deg
Lon = 2 cm



sPHENIX20
Grid size:
Rad = 0.75 cm
Phi = 360 deg
Lon = 0.64 cm



sPHENIX30
Grid size:
Rad = 0.63 cm
Phi = 360 deg
Lon = 0.64 cm

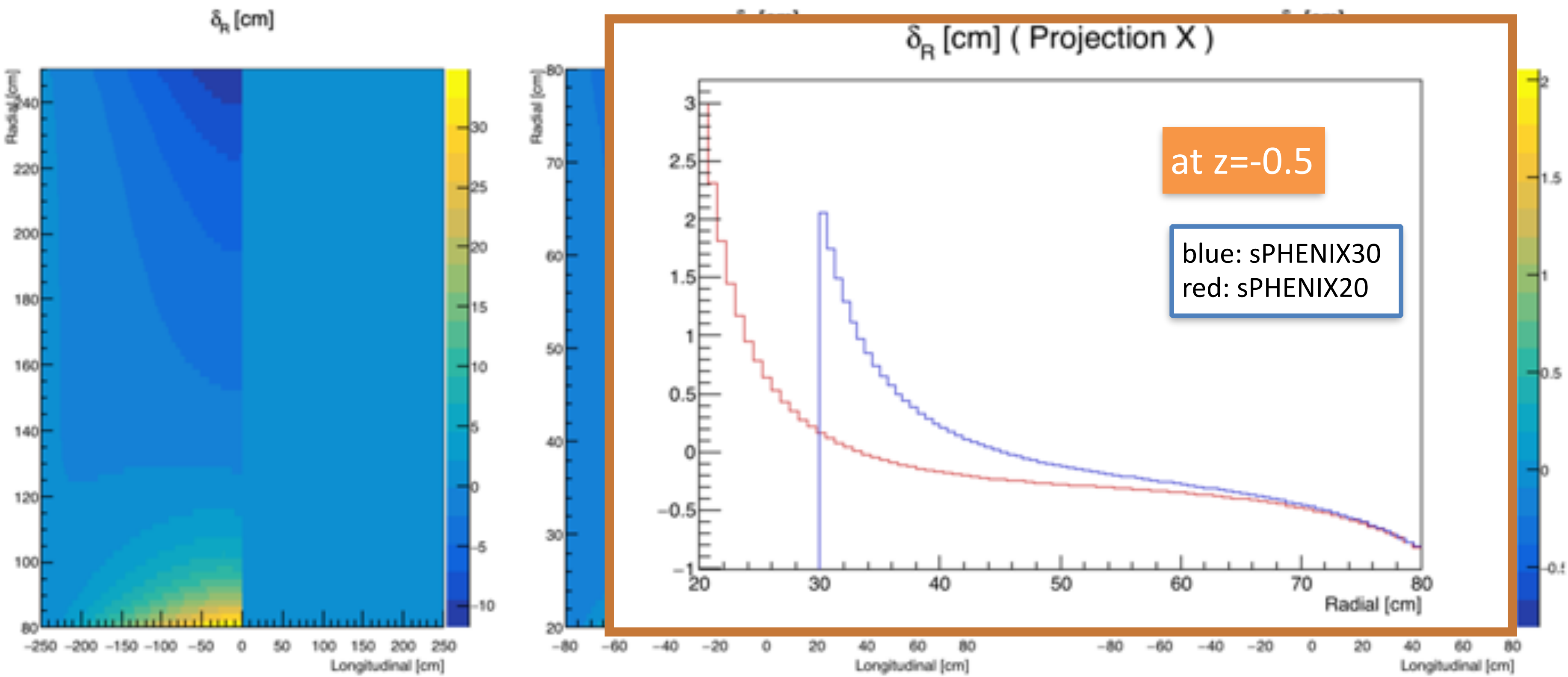


Estimated mean distortions in R

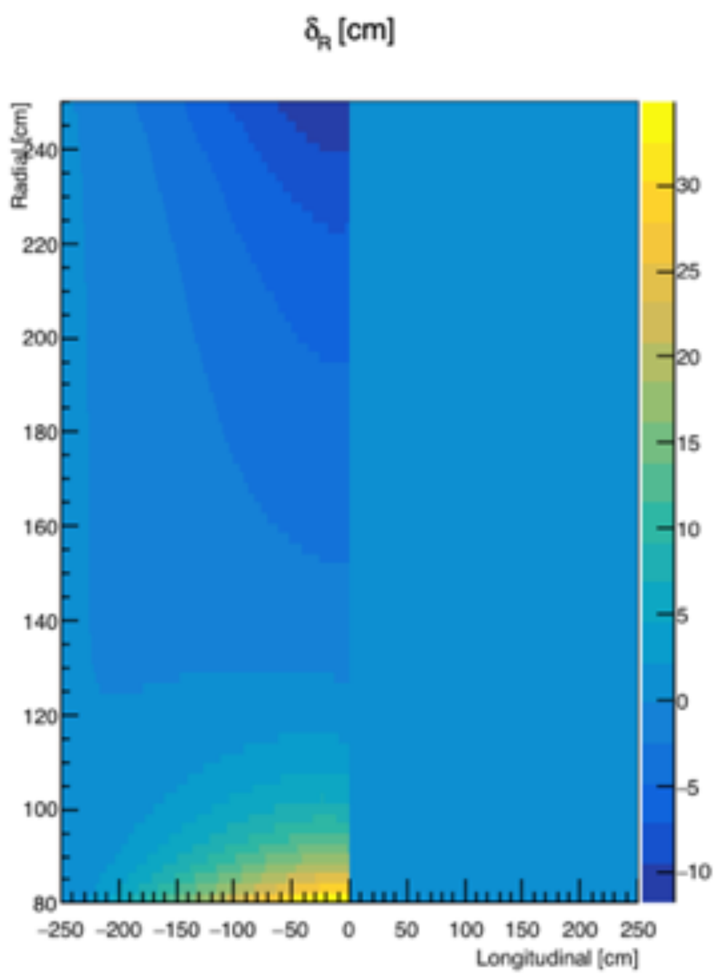
ALICE
Grid size:
Rad = 2.13 cm
Phi = 360 deg
Lon = 2 cm

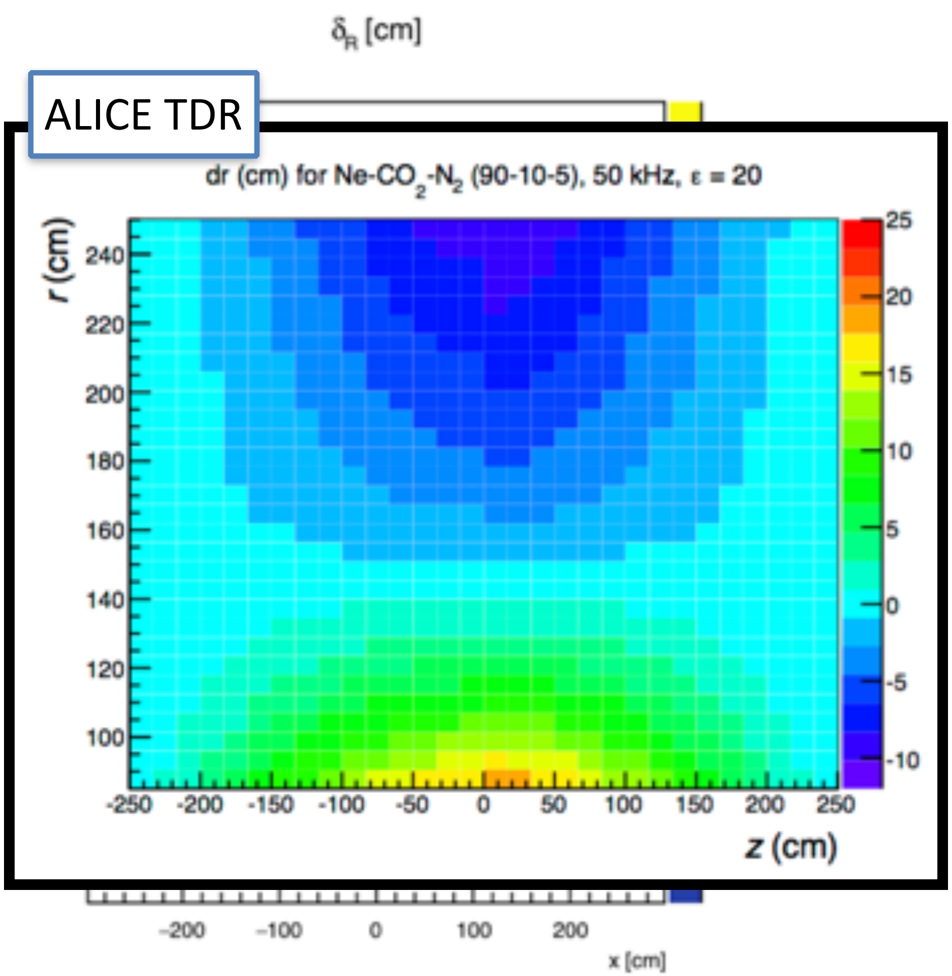
sPHENIX20
Grid size:
Rad = 0.75 cm
Phi = 360 deg
Lon = 0.64 cm

sPHENIX30
Grid size:
Rad = 0.63 cm
Phi = 360 deg
Lon = 0.64 cm

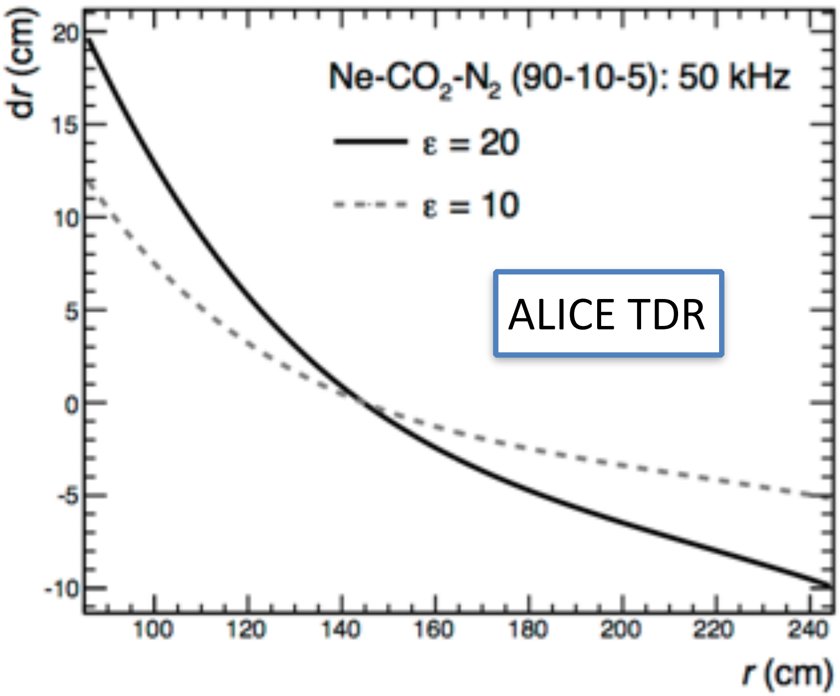


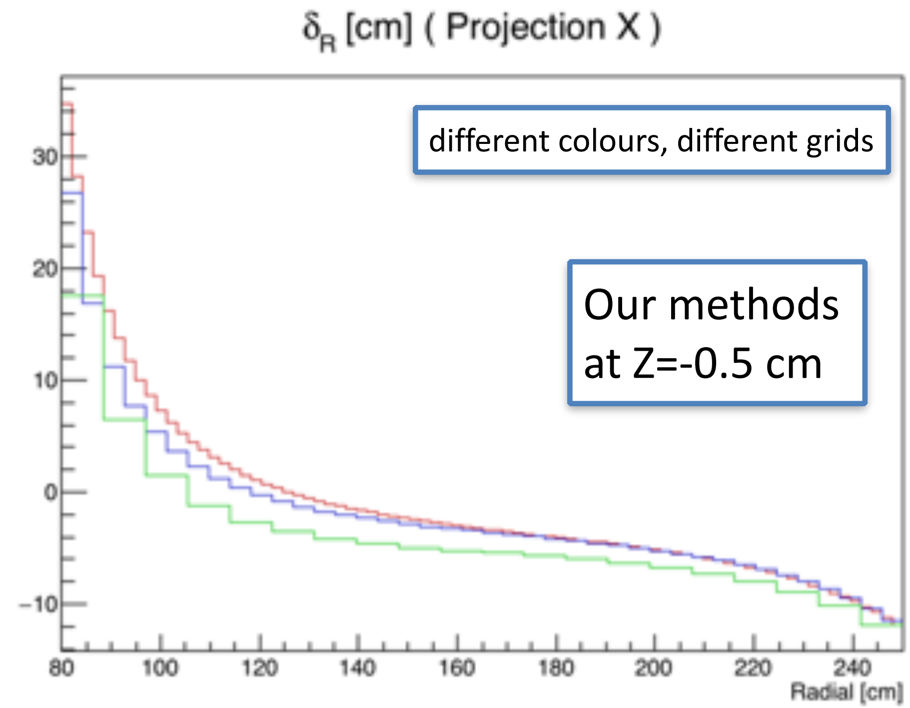
Comparing with ALICE TDR (1/2)





Comparing with ALICE TDR (2/2)



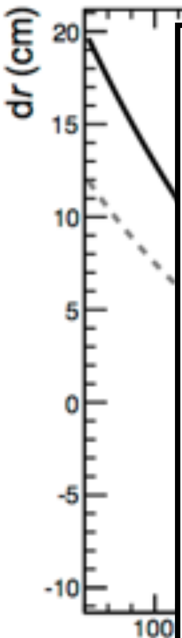


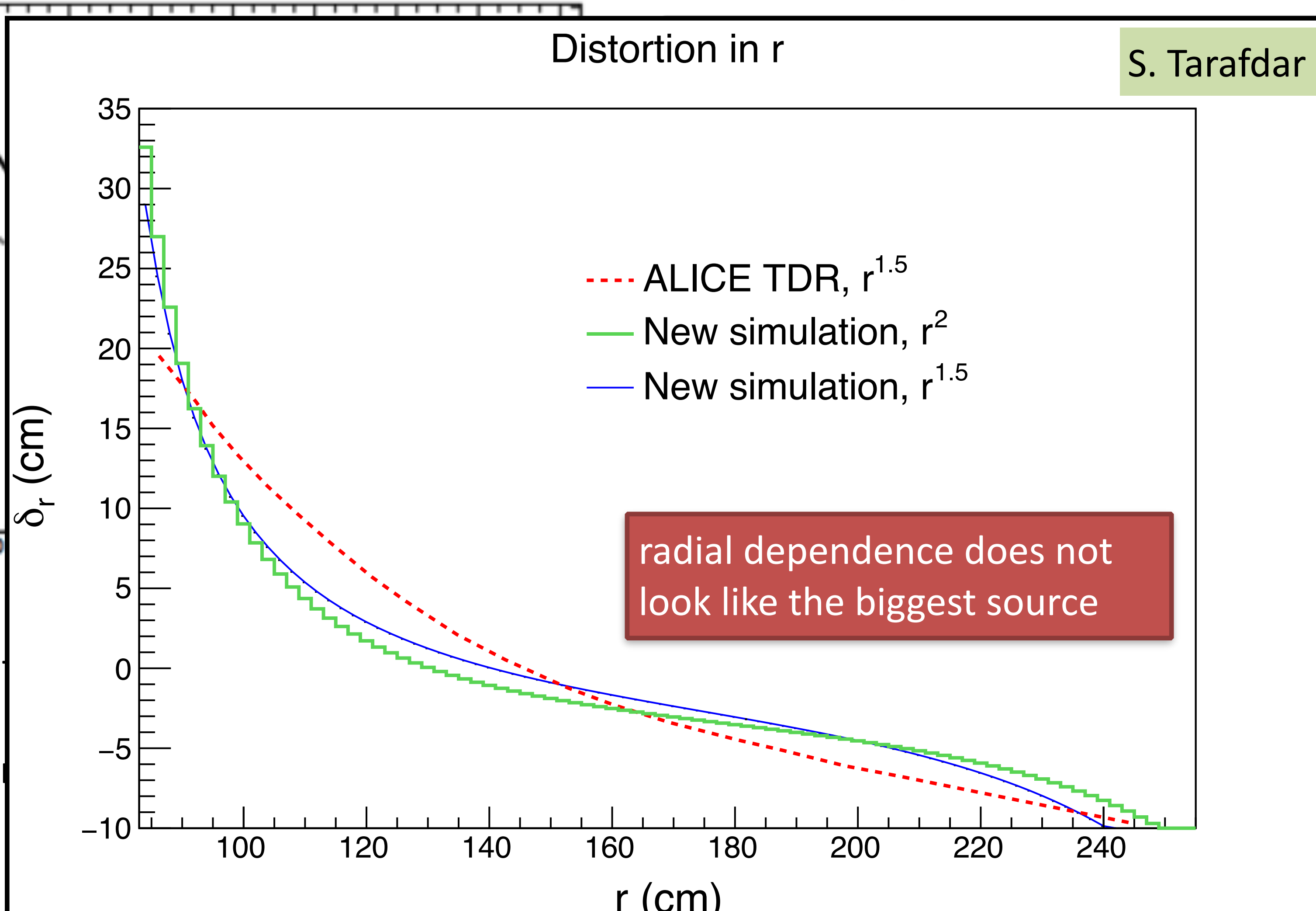
Quantitatively close, but not quite the right shape

Source of incongruence:

- We do Laplace expansion up to 15th order (ALICE claims 30th)
- We use Ez = E0 + dEz (ALICE does not say)
- We probe Dr at z=-0.5 cm (ALICE gets it at z=0.5)
- We use 1/r^2 in ICD (ALICE claims 1/r^1.5)
- …?

Comparing with ALICE TDR (2/2)





δ_R [cm] (Projection X)

Quanti

Incongr

0th)

- …?

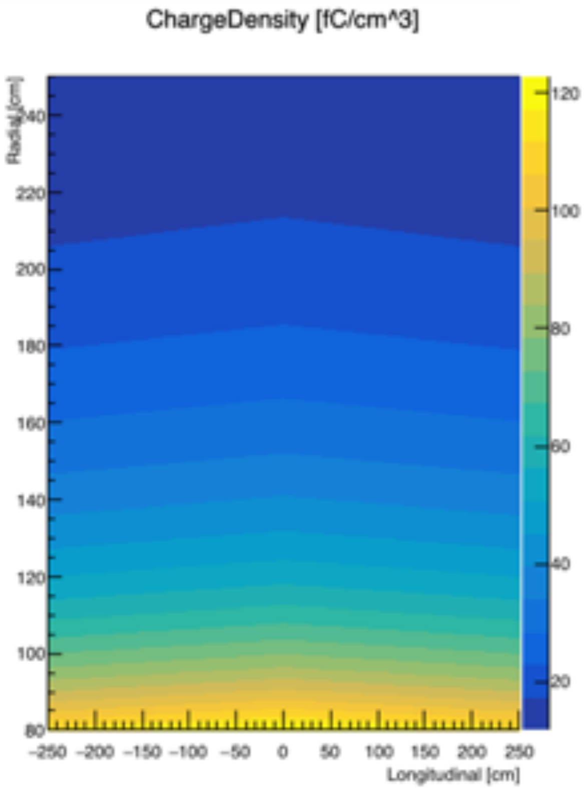
Initial Charge Density

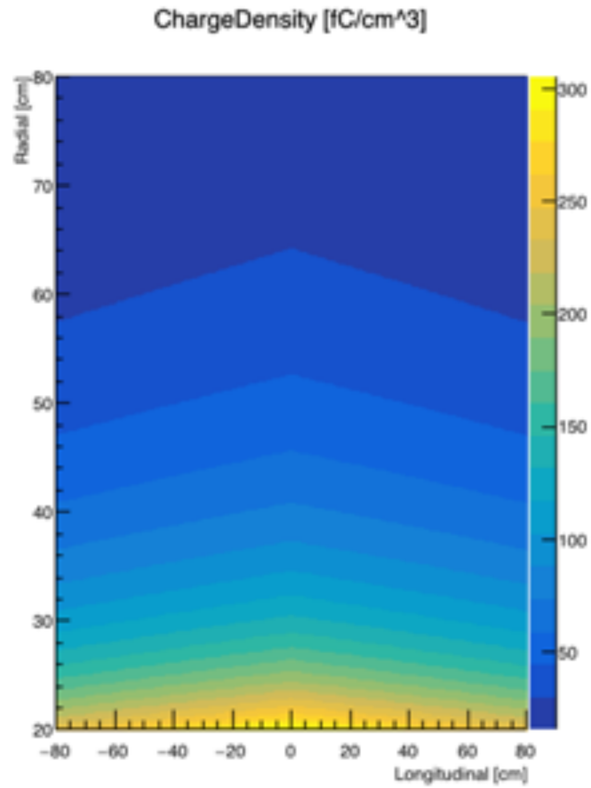
ALICE
Radial dependence set at 2
Gas factor at 1.0/76628.0
Multiplicity at 900
DC Rate at 50kHz
BackFlow at 20 (=1.0%2000)

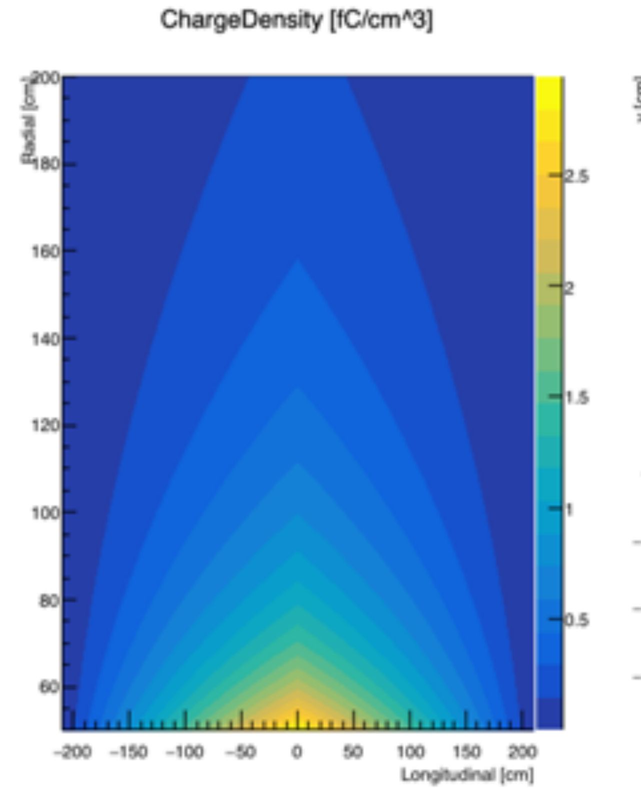
sPHENIX20
BackFlow at 6 (=0.3%2000)

A better description of STAR case is under investigation. See Sourav's slides in backup.

STAR
Radial dependence set at 2
Gas factor at 1.0/76628.0
Multiplicity at 450
DC Rate at 15kHz
BackFlow at 0





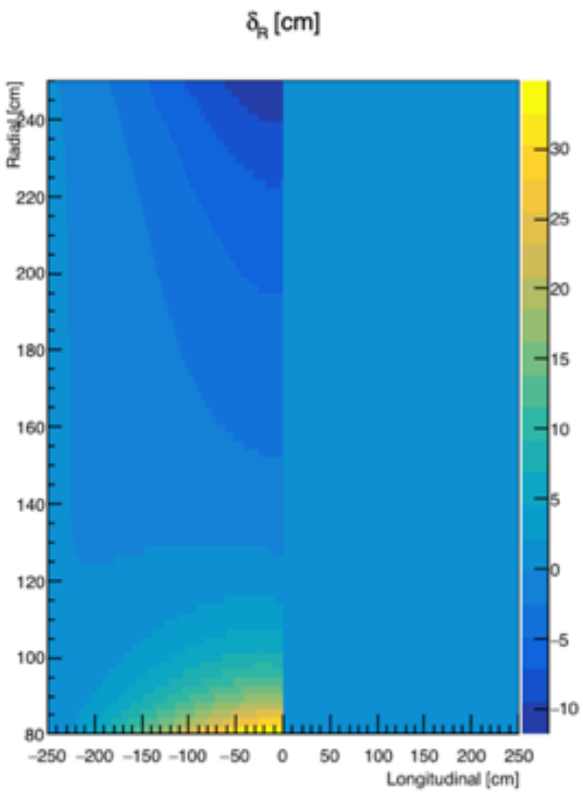


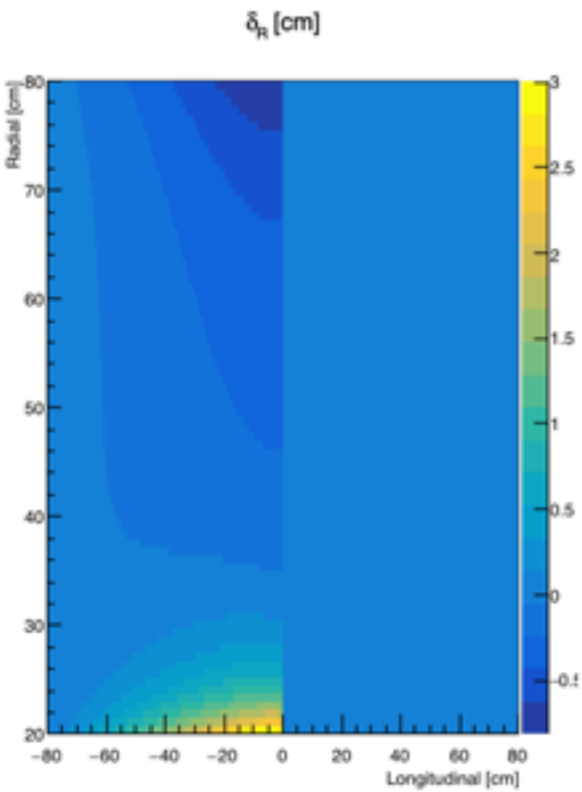
Estimated mean distortions in R

ALICE
Grid size:
Rad = 2.13 cm
Phi = 360 deg
Lon = 2 cm

sPHENIX20
Grid size:
Rad = 0.75 cm
Phi = 360 deg
Lon = 0.64 cm

STAR
Grid size:
Rad = 1.88 cm
Phi = 360 deg
Lon = 1.68 cm





work in progress

An step forward on ICD

- Initial Charge Density was modelled so far using phenomenological expression from ALICE/STAR

- Many control variables like “gas factor”, “multiplicity”, “ion-feedback” are used heuristically.

- To gain full control on the gas response and realistic track density, it is desirable to model this from First Principles.

Algorithm Flow chart

ChargeMap(X)

1. Contains list of ions/electrons in TPC

RecordTime

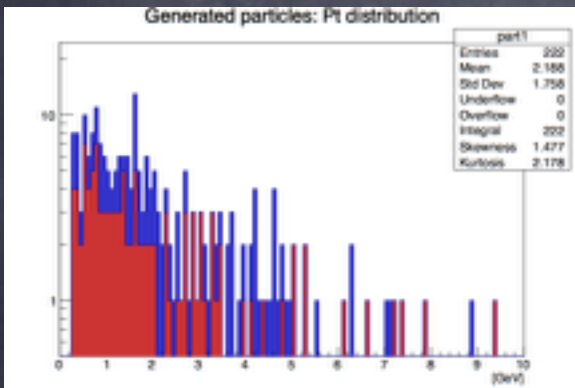
Event

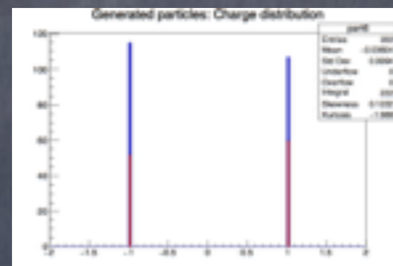
1. Generate particles (X,P)
2. Particles -> helix traces
3. traces -> electron - ion
4. pushes new pairs into "ChargeMap"

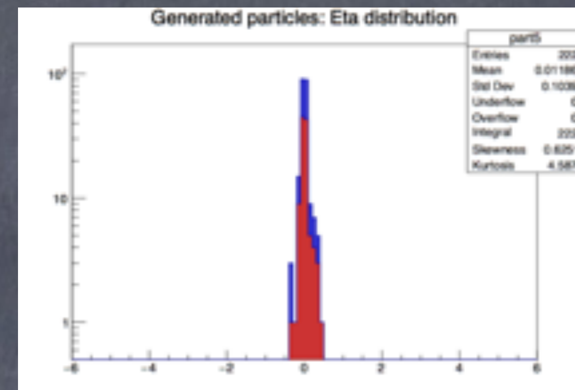
Transport

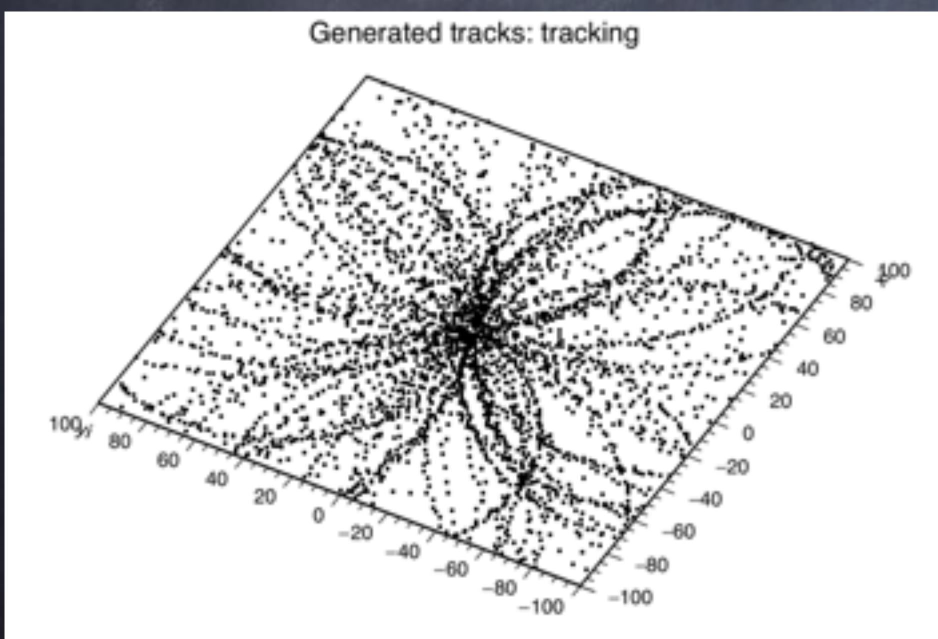
1. Evolves ChargeMap in lapse between events

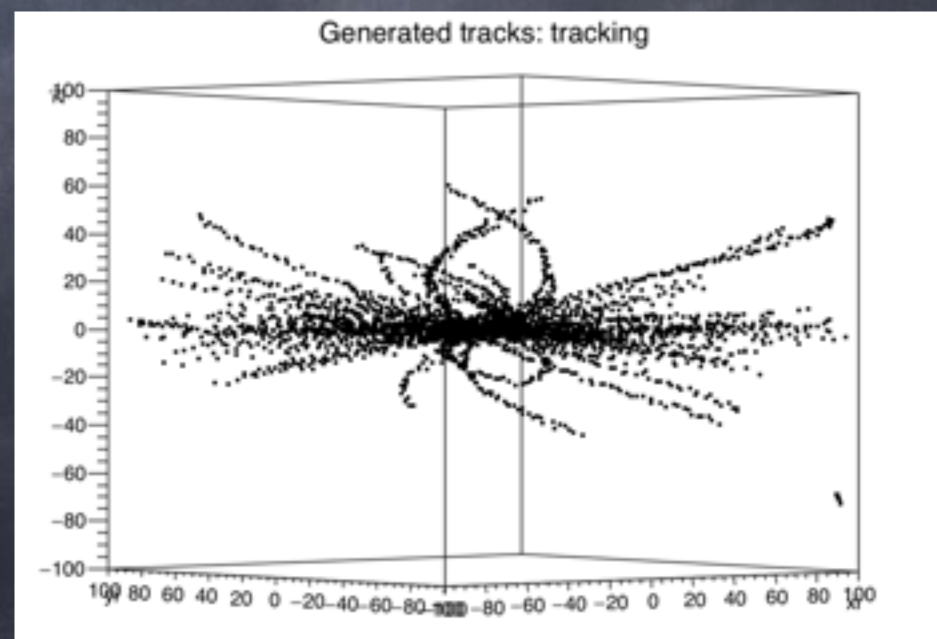
Tracing particles in a toy event











Traces to pairs

- Ingredients
  - DeltaE for the total track length
  - DeltaE to N ionised electrons

Gas	Ratio	Density*10^{-3} (g/cm^3)	Radiation Length (m)	N_p (cm^{-1})	N_t (cm^{-1})
Ne-CH_4	90-10	0.881	361.8	13.45	44
	80-20	0.862	380.4	14.9	45
	70-30	0.843	401	16.35	46
Ne-C_2H_6	90-10	.0944	344	14.9	49.8
	80-20	0.988	343.9	17.8	56.6
	70-30	1.032	343.4	20.7	63.4
Ne-iC_4H_{10}	90-10	1.06	312	19.2	58.2
	80-20	1.23	285	26.4	73.4
	70-30	1.4	262	33.6	88.6
Ne-CO_2	90-10	1	317	14.35	47.8
	80-20	1.12	293	16.7	52.6
	70-30	1.22	272	19	57.4
Xe-CH_4	90-10	5.34	16.6	42.25	281.6
	80-20	4.83	18.6	40.5	256.2
	70-30	4.31	21.2	38.75	230.8
Xe-C_2H_6	90-10	5.4	16.6	43.7	287.4
	80-20	4.95	18.5	43.4	267.8
	70-30	4.5	21	43.1	248.2
Xe-iC_4H_{10}	90-10	5.53	16.5	48	295.8
	80-20	5.2	18.3	52	284.6
	70-30	4.87	20.6	56	273.4
Xe-CO_2	90-10	5.47	16.5	43.15	285.4
	80-20	5.1	18.4	42.3	263.8
	70-30	4.69	20.7	41.45	242.2

Table 1. (Continued) Parameters of some gas and gas mixtures.

For the moment, I parametrised the number of Nt per cm as cte from this table

Few tasks still ahead

- Connect particle pool to generator(s)
- Improve characterisation of ionisation in gas
- ...

BACKUP

Space charge density for STAR

$$\rho\,(r\_,\ z\_) := A\left(\frac{1 - b\ z + c\ \epsilon}{f_d\ r^d}\right)$$

- Inner radius = 50 cm, Outer radius = 200 cm, Longitudinal length = 210 cm, phi = 2*pi
- A = [G] X [m] X [r]X [e_0] /19318. [in C/m]
  - e_0 (=8.85e-12) : vacuum permittivity
  - G = 1
  - M : Event multiplicity = 170
  - R : Total interaction rate = 15 kHz
- b : 1/driftlength
- c*e : 0 for all the plots in next few slides
- Radial dependence (d) = 2 & f_d =1

ALICE has this factor named "empirical factor" quoted as 76628

Derivation for ALICE from STAR

3.2. Scaling STAR observations to ALICE expectations

The normalized distribution of charge density used in the STAR TPC to correct for the space-charge effect is:

$$\rho(r,z) = \frac{(L-z)}{L}\frac{(r_O^2 - r_I^2)}{\log(r_O/r_I)}\frac{0.01}{1.5\cdot 10^6}\frac{IR}{r^2} \qquad (6)$$

where r_O=200 cm and r_I=47.9 cm are the outer and inner radii (STAR TPC dimensions). The empirical factor which corresponds roughly to an interaction rate (IR) of 15 kHz for Au–Au collisions at a center of mass energy of 200 GeV is then:

$$F_E = \frac{(r_O^2 - r_I^2)}{\log(r_O/r_I)}\frac{0.01}{1.5\cdot 10^6} = 1.76\cdot 10^{-2}$$

Using formula (6), we can calculate an empirical factor for the ALICE TPC where we include a scaling factor for the min.bias multiplicities (with $M_{mb,S} = 127$ for the top 80 % within STAR) and the design scaling factor F_D from Tab. 4:

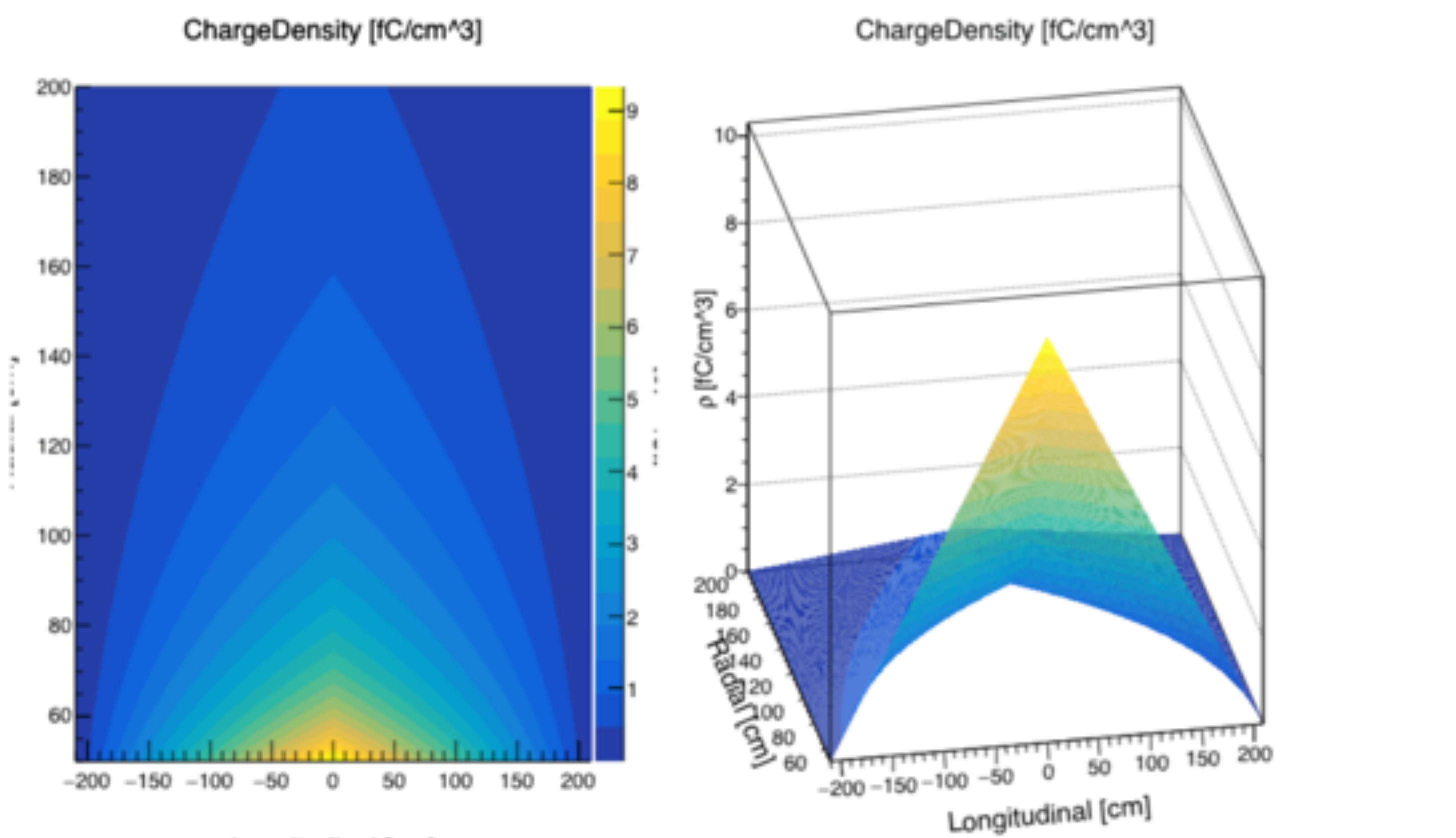
$$F_{E,A} = (F_E \cdot F_D / M_{mb,S})^{-1} = 76628\,.$$

The complete empirical formula can be written as:

- **So for STAR the empirical factor should look like :**
  $F_{E,S}$ = (Min bias multiplicity in STAR) / F_E , where F_e 1.76 e-02
  = 340./ 1.76e-02 = 19318.2

Few cases of STAR charge density by changing the gas factor and keeping c*eps = 0

Gas Factor [G] = 1.0



$$\rho(50\mathrm{cm}, 0\mathrm{cm}) = 1.9 \times 10^{-14} \frac{\mathrm{coul}}{\mathrm{cm}^3}$$

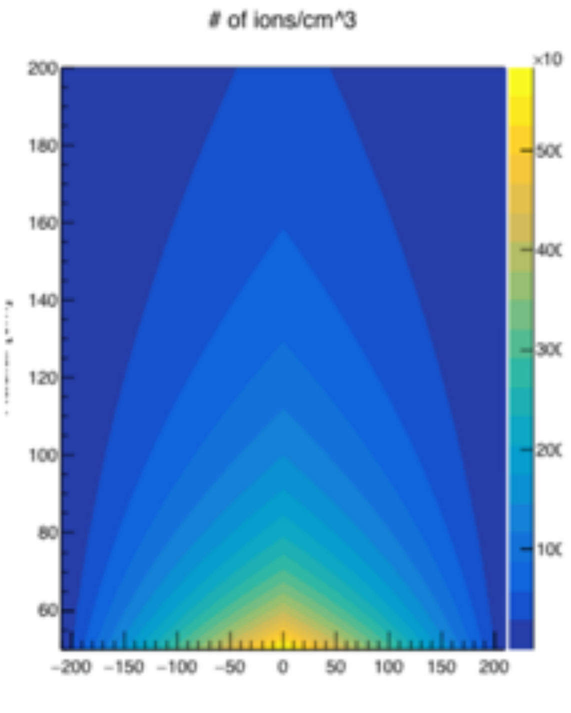
$$\rho(50\mathrm{cm}, 0\mathrm{cm}) = 1.2 \times 10^{5} \frac{\mathrm{qe}}{\mathrm{cm}^3}$$

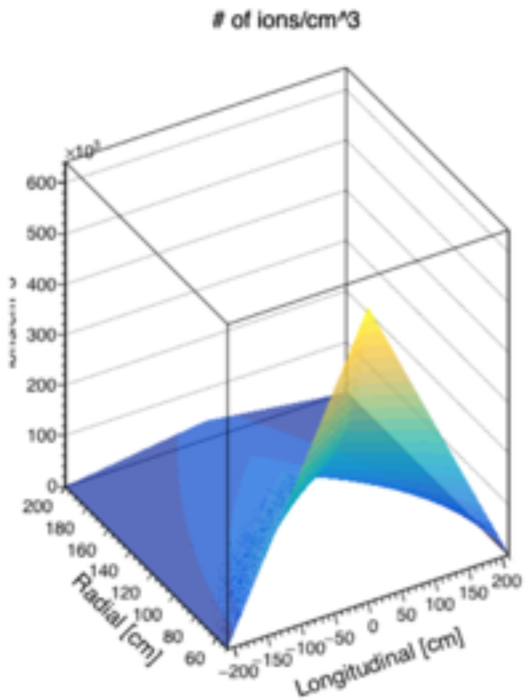
STAR values

Charge density in new simulation using Toy model ~ 8.5 e-14 C/cm^3 ~ 5.3e+05 qe/cm^3

STAR number of ions density with gas factor [G] = 1.0

New simulation





Number of ions per cm^3 using new simulation ~ 5.3e+05 qe/cm^3

STAR estimate

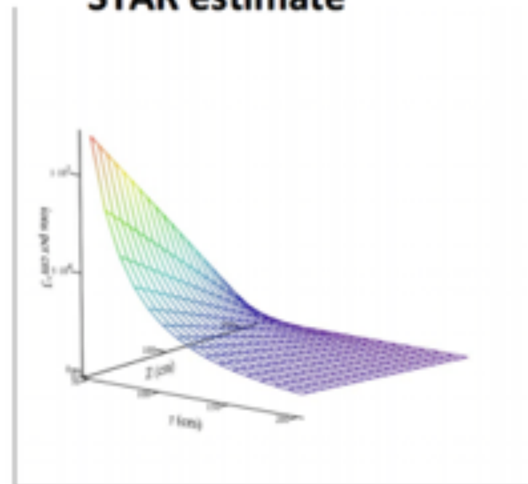


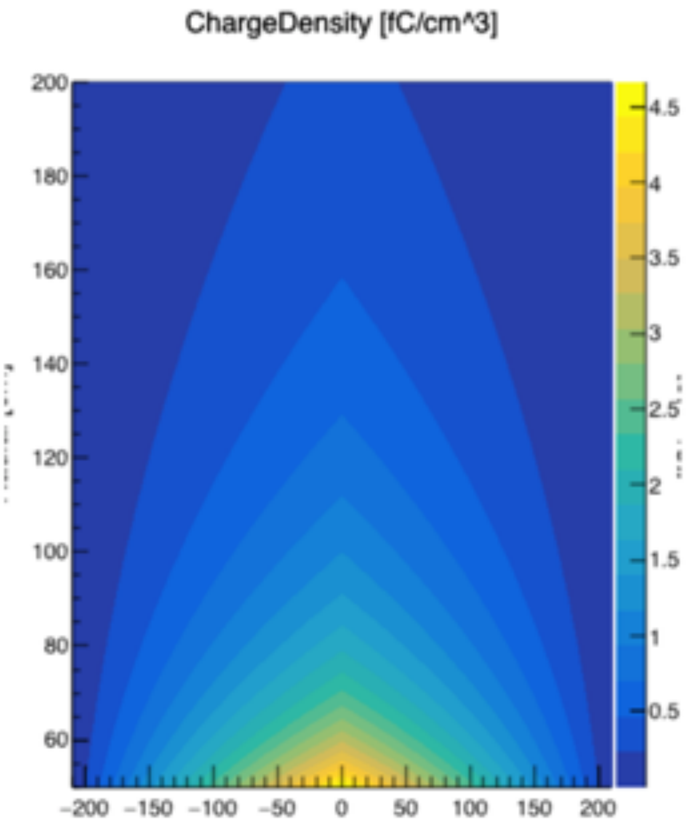
Fig. 2 Steady state positive ion density as a function of r and z in the TPC volume for RHIC II predicted AuAu luminosity. This is the positive ion density that produced the distortion shown in Fig. 1

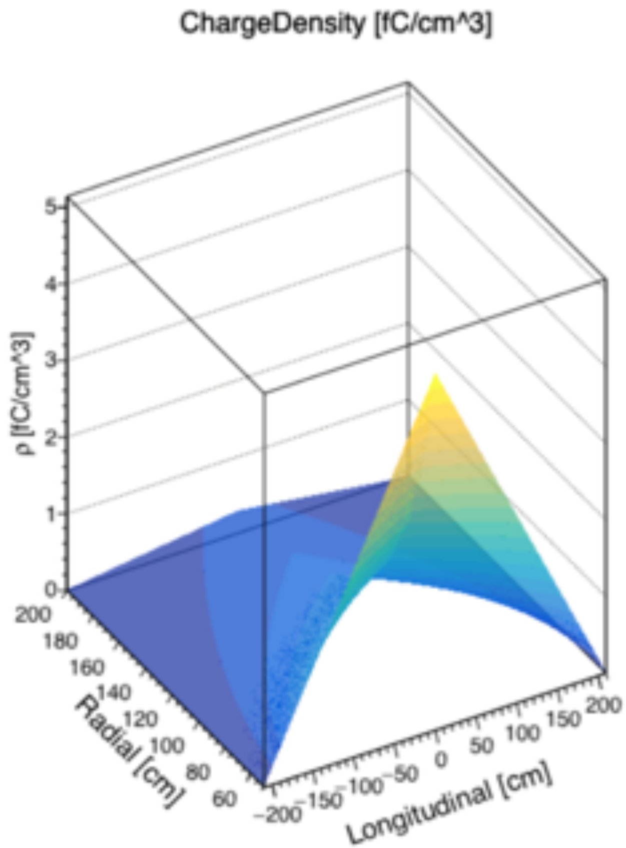
STAR values

$$\rho(50\text{cm}, 0\text{cm}) = 1.9 \times 10^{-14} \frac{\text{coul}}{\text{cm}^3}$$

$$\rho(50\text{cm}, 0\text{cm}) = 1.2 \times 10^{5} \frac{\text{qe}}{\text{cm}^3}$$

STAR charge density with gas factor [G] = 0.5





STAR values

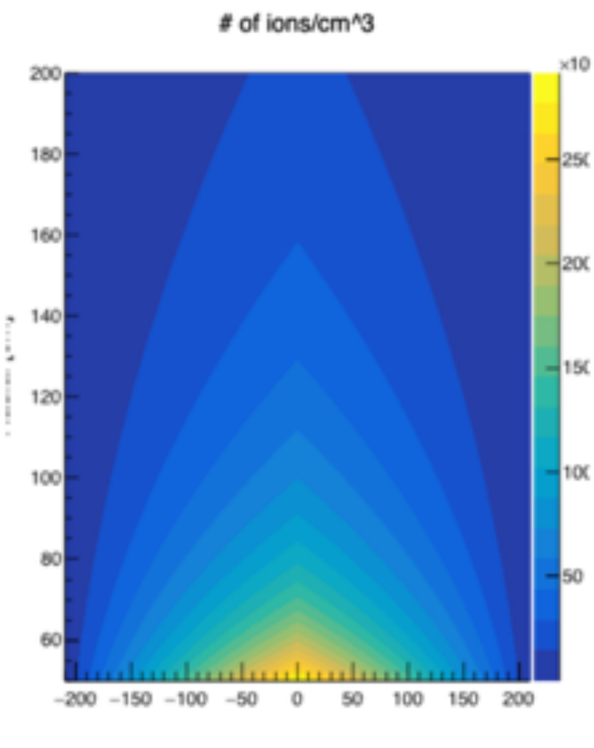
$$\rho(50\text{cm}, 0\text{cm}) = 1.9 \times 10^{-14} \frac{\text{coul}}{\text{cm}^3}$$

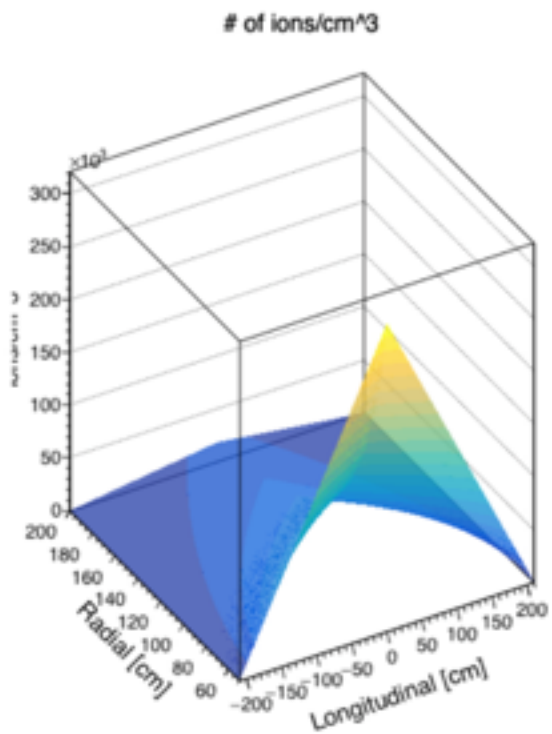
$$\rho(50\text{cm}, 0\text{cm}) = 1.2 \times 10^{5} \frac{\text{qe}}{\text{cm}^3}$$

Charge density using new simulation
~ 4.5 e-14 C/cm^3

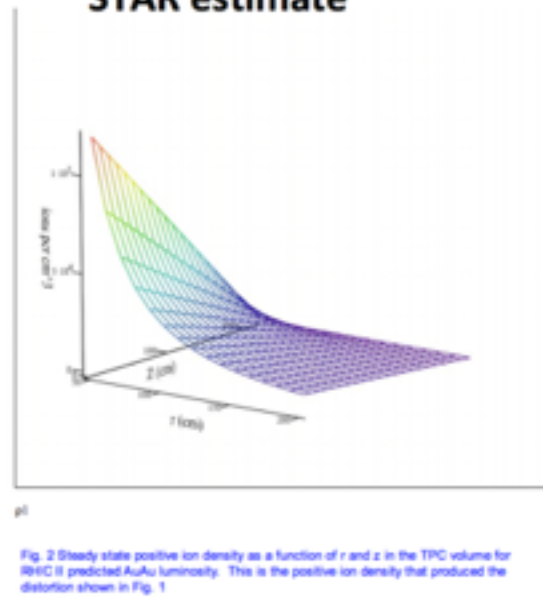
STAR number of ions density with gas factor [G] = 0.5

New simulation





STAR estimate



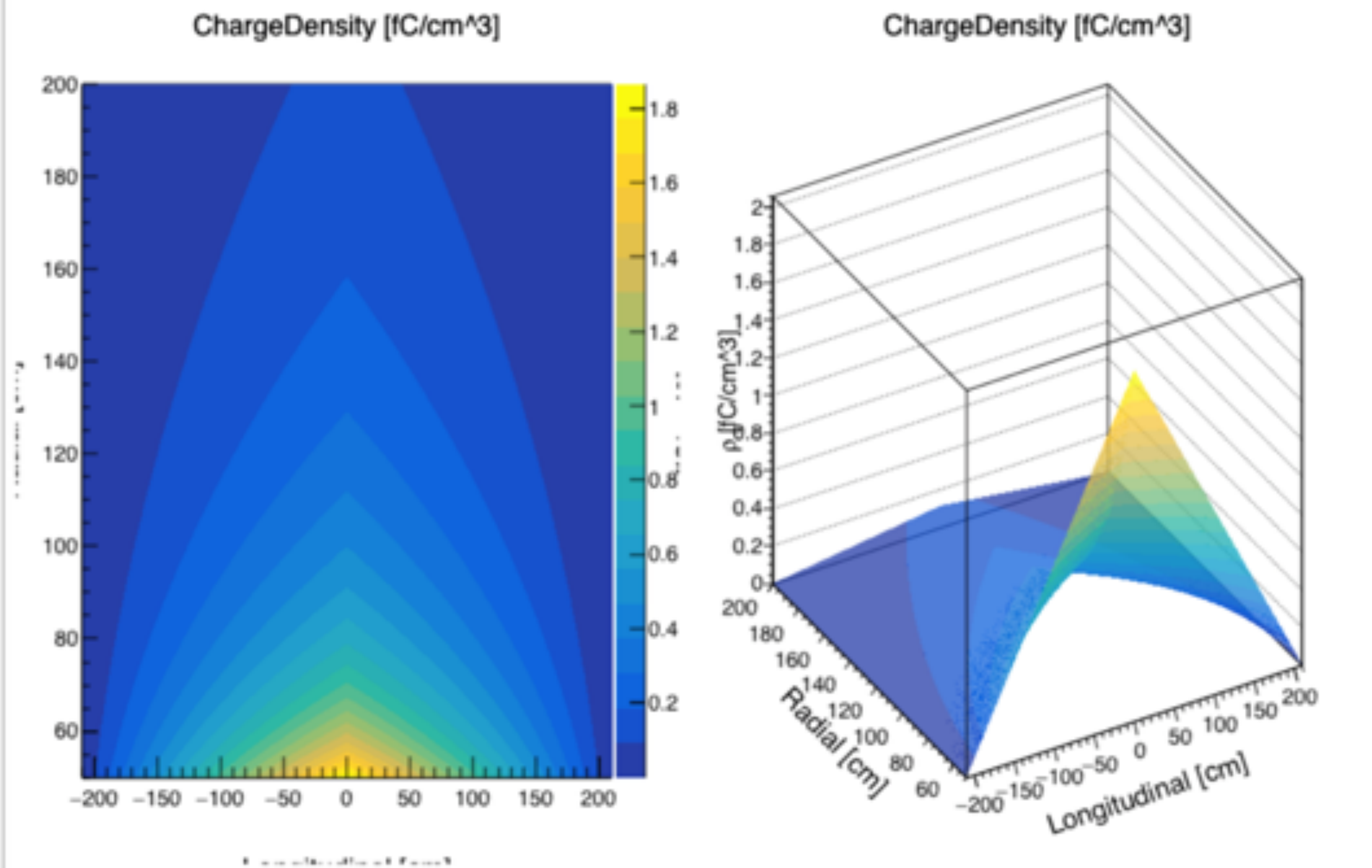
Charge density using new simulation
~ 2.8 e+05 qe/cm^3

STAR values

$$\rho(50\text{cm},0\text{cm}) = 1.9\times10^{-14}\ \frac{\text{coul}}{\text{cm}^3}$$

$$\rho(50\text{cm},0\text{cm}) = 1.2\times10^{5}\ \frac{\text{qe}}{\text{cm}^3}$$

STAR charge density with gas factor [G] = 0.2



STAR values

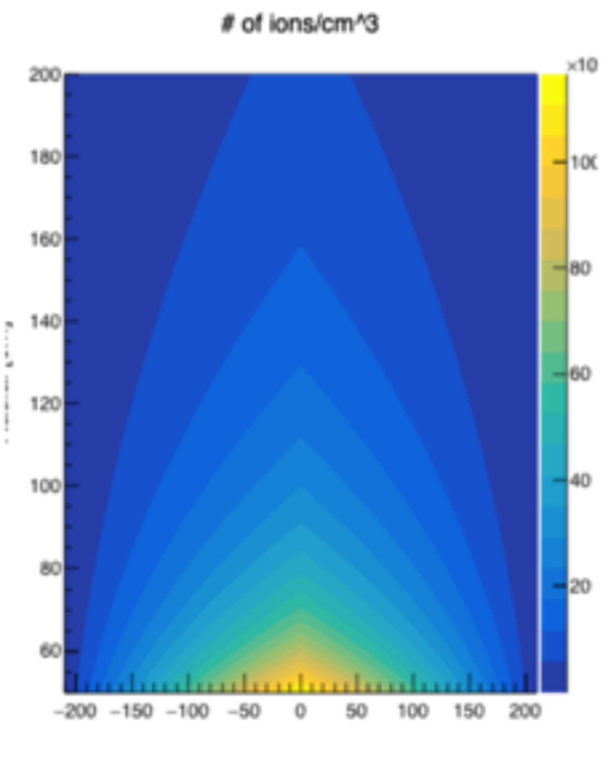
$$\rho(50\mathrm{cm}, 0\mathrm{cm}) = 1.9 \times 10^{-14}\,\frac{\mathrm{coul}}{\mathrm{cm}^3}$$

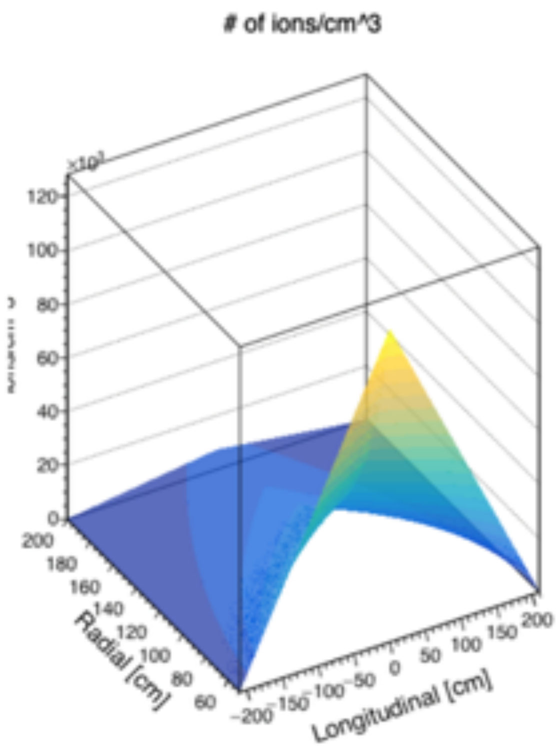
$$\rho(50\mathrm{cm}, 0\mathrm{cm}) = 1.2 \times 10^{5}\,\frac{\mathrm{qe}}{\mathrm{cm}^3}$$

Charge density using new simulation
~ 1.8 e-14 C/cm^3

STAR number of ions density with gas factor [G] = 0.2

New simulation





STAR estimate

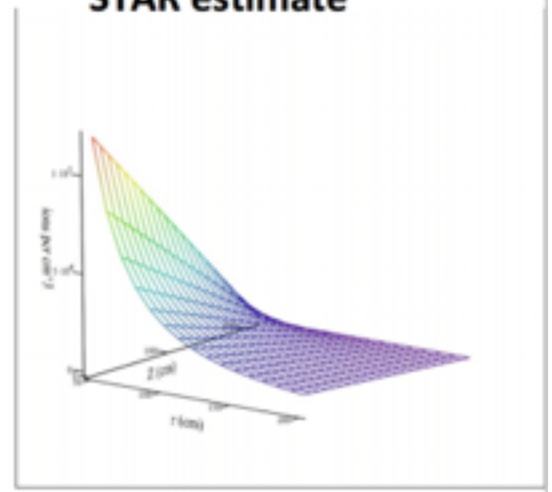


Fig. 2 Steady state positive ion density as a function of r and z in the TPC volume for RHIC II predicted AuAu luminosity. This is the positive ion density that produced the distortion shown in Fig. 1

STAR values

$$\rho(50\mathrm{cm}, 0\mathrm{cm}) = 1.9\times 10^{-14}\,\frac{\mathrm{coul}}{\mathrm{cm}^3}$$

$$\rho(50\mathrm{cm}, 0\mathrm{cm}) = 1.2\times 10^{5}\,\frac{\mathrm{qe}}{\mathrm{cm}^3}$$

Charge density using new simulation
~ 1.12 e+05 qe/cm^3

Setting gas factor [G] between 0.2 and 0.5 gives almost same estimate of space charge as done by STAR for STAR TPC

Space charge density in the TPC volume

Toy model:

$$\rho(r\_, z\_) := A \left(\frac{1 - b\, z + c\, \epsilon}{f_d\, r^d} \right)$$

1. Proportionality to the primary ionisation (i.e. local track density in a collision) r^-2 dependence and Z drift velocity
2. Back flow dependence as CTE in Z direction

Space charge density in the TPC volume

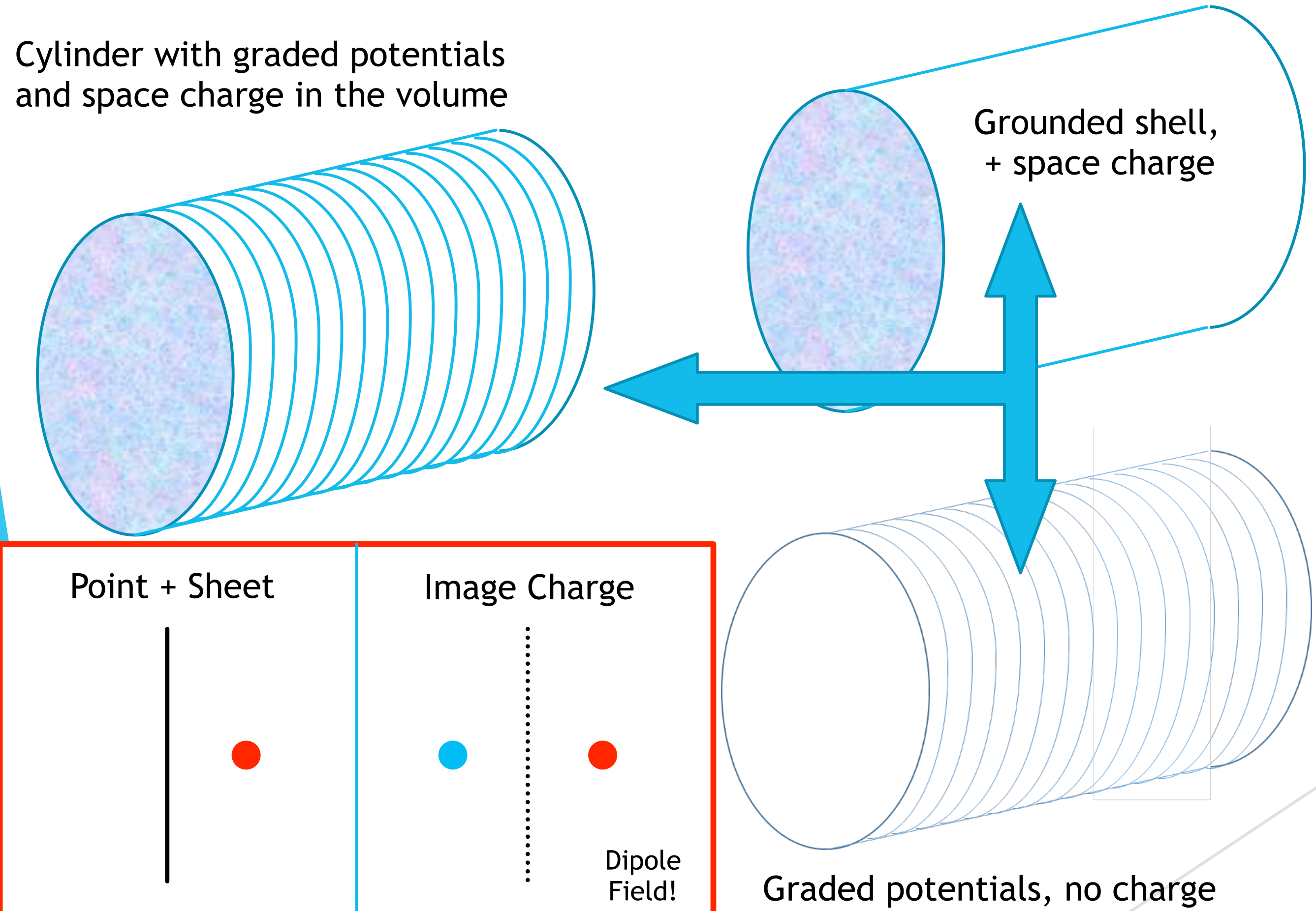
$$\rho(r\_, z\_) := A\left(\frac{1 - b\, z + c\, \epsilon}{f_d\, r^d}\right)$$

- A = [G] x [M] x [R] x [e_0] / 76628   [in C/m]
  - e_0 (=8.85e-12): vacuum permittivity [in As/(Vm)]
  - G (=1): gas factor (prim ioniz. / drift velocity)
  - M (=950): nominal event multiplicity
  - R (=5e4): total interaction rate [in Hz]
- b (=1/2.5): 1/DriftLength [in 1/m]
- c*e (=2/3*20)
- d (=2 for STAR f_d=1; =1.5 for ALCE)

All gas parameters are embedded in G/76628

Tom

Factorization of the Space Charge Problem



- Graded field cage field determined by ANSYS or COLSOL finite element calculations.
- Grounded shell solved using Greene's theorem

$$\Delta G(r, r_{ch}) = \delta(r - r_{ch})$$

$$E_{ch}(r, r_{ch}) = \nabla G(r, r_{ch})$$

$$E = \int \rho(r_{ch}) E_{ch}(r, r_{ch}) dV_{ch}$$

Carlos Tom

Tom

Basic Approach to Solving the Cylinder

- The problem at hand is this: $\Delta G(\vec{x};\vec{x}') = -\delta(\vec{x}-\vec{x}')$, (5.13)

$$\left[\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}\right]G(r,\phi,z;r',\phi',z') = -\frac{\delta(r-r')}{r'}\,\delta(\phi-\phi')\,\delta(z-z'), \quad (5.14)$$

- Our solution begins with solving the homogeneous equation to provide a basis set of functions for the full solution:

$$\Delta\Phi = 0, \qquad \left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}\right)\Phi(r,\phi,z) = 0, \qquad \Phi(r,\phi,z) = R(r)\Phi(\phi)Z(z).$$

Periodicity set m=0,1,2,3,…

$$\Phi_m(\phi) = C_m\, e^{im\phi} = A_m\cos(m\phi) + B_m\sin(m\phi) \qquad \text{with } m\in\mathbb{Z}.$$

$$\frac{R_{rr}}{R}+\frac{1}{r}\frac{R_r}{R}-\frac{m^2}{r^2} = -\frac{Z_{zz}}{Z} = \begin{cases} -\beta^2, & \text{case I;} \\ \beta^2, & \text{case II.} \end{cases}$$

Solution without boundary conditions applied:

$$Z_m(z) = C_m\cosh(\beta z) + D_m\sinh(\beta z),$$
$$R_m(r) = E_m J_m(\beta r) + F_m Y_m(\beta r).$$

Constants formulated to explicitly vanish at r=a

$$R_{mn}(r) = Y_m(\beta_{mn}a)J_m(\beta_{mn}r) - J_m(\beta_{mn}a)Y_m(\beta_{mn}r).$$

Vanishing at r=b forces β to become discreet.

Tom

Finishing the solution

- Once the solutions to the homogeneous equation are known, we express the Dirac delta function in this basis:

$$\begin{aligned}
\delta(\phi-\phi') &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = \frac{1}{2\pi}\sum_{m=0}^{\infty}(2-\delta_{m0})\cos[m(\phi-\phi')], \\
\frac{\delta(r-r')}{r} &= \sum_{n=1}^{\infty}\frac{R_{mn}(r)R_{mn}(r')}{\bar{N}_{mn}^2} \quad \text{with} \quad \bar{N}_{nm}^2 = \int_a^b R_{mn}^2(r)\, r dr, \\
m &= 0, 1, 2, \dots .
\end{aligned}$$

- After which the solution is readily obtained:

$$G(r,\phi,z;r',\phi',z') = \frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(2-\delta_{m0})\cos[m(\phi-\phi')]\frac{R_{mn}(r)R_{mn}(r')}{\bar{N}_{mn}^2}\frac{\sinh(\beta_{mn}z_<)\sinh(\beta_{mn}(L-z_>))}{\beta_{mn}\sinh(\beta_{mn}L)}.$$

- Although the solution is correct, it is not assured to be readily convergent.
- Rossegger used three independent basis sets to obtain stable, differentiable, convergent solutions for the r, ϕ, and z components of the field:

$$\frac{\partial}{\partial z}G(r,\phi,z,r',\phi',z') = \frac{1}{2\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(2-\delta_{m0})\cos[m(\phi-\phi')]\frac{R_{mn}(r)R_{mn}(r')}{N_{mn}^2}\frac{\partial}{\partial z}\left(\frac{\sinh(\beta_{mn}z_<)\sinh(\beta_{mn}(L-z_>))}{\beta_{mn}\sinh(\beta_{mn}L)}\right), \tag{5.64}$$

with $$\frac{\partial}{\partial z}\left(\sinh(\beta_{mn}z_<)\sinh(\beta_{mn}(L-z_>))\right) = \begin{cases} \beta_{mn}\cosh(\beta_{mn}z)\sinh(\beta_{mn}(L-z')), & \text{for } 0 \le z < z' \le L, \\ -\beta_{mn}\cosh(\beta_{mn}(L-z))\sinh(\beta_{mn}z'), & \text{for } 0 \le z' < z \le L. \end{cases}$$

$$\frac{\partial}{\partial r}G(r,\phi,z,r',\phi',z') = \frac{1}{\pi L}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(2-\delta_{m0})\cos[m(\phi-\phi')]\sin(\beta_n z)\sin(\beta_n z')\frac{\partial}{\partial r}\left(\frac{R_{mn\,1}(r_<)R_{mn\,2}(r_>)}{I_m(\beta_n a)K_m(\beta_n b)-I_m(\beta_n b)K_m(\beta_n a)}\right), \tag{5.65}$$

with $$\frac{\partial}{\partial r}\left(R_{mn\,1}(r_<)R_{mn\,2}(r_>)\right) = \begin{cases} R'_{mn}(a,r)R_{mn\,2}(r'), & \text{for } a \le r < r' \le b, \\ R_{mn\,1}(r')R'_{mn}(b,r), & \text{for } a \le r' < r \le b, \end{cases}$$

wherein $R'_{mn}(s,t)$ is

$$R'_{mn}(s,t) = \frac{\beta_n}{2}\left(K_m(\beta_n s)(I_{m-1}(\beta_n t)+I_{m+1}(\beta_n t)) + I_m(\beta_n s)(K_{m-1}(\beta_n t)+K_{m+1}(\beta_n t)\right).$$

$$\frac{\partial}{\partial \phi}G(r,\phi,z,r',\phi',z') = \frac{1}{L}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sin(\beta_n z)\sin(\beta_n z')\frac{R_{nk}(r)R_{nk}(r')}{N_{nk}^2}\frac{\partial}{\partial \phi}\left(\frac{\cosh[\mu_{nk}(\pi-|\phi-\phi'|)]}{\mu_{nk}\sinh(\pi\mu_{nk})}\right) \tag{5.66}$$

with $$\frac{\partial}{\partial \phi}\left(\cosh[\mu_{nk}(\pi-|\phi-\phi'|)]\right) = \begin{cases} -\mu_{nk}\sinh[\mu_{nk}(\pi-(\phi-\phi')), & \text{for } 0 \le \phi' < \phi \le 2\pi \\ \mu_{nk}\sinh[\mu_{nk}(\pi-(\phi'-\phi)), & \text{for } 0 \le \phi < \phi' \le 2\pi \end{cases}$$

Langevin Eq:

charge of the drifting particle

Friction (K>0)

$$m \frac{d\vec{u}}{dt} = q e \vec{E} + q e \left[\vec{u} \times \vec{B}\right] - K \vec{u}$$

drift velocity

EB force

Solution:

$t \gg m / K$ Adiabatic approx.

$\frac{d\vec{u}}{dt} = 0$ Steady state

$$\vec{u} = \frac{\mu \left|\vec{E}\right|}{1 + \omega^2 \tau^2} \left[\hat{E} + \omega \tau \left(\hat{E} \times \hat{B}\right) + \omega^2 \tau^2 \left(\hat{E} \cdot \hat{B}\right) \hat{B}\right]$$

scalar mobility of the electric field

$\mu = \frac{q e}{K}$

mean interaction time between drifting electrons and atoms from the gas

cyclotron frequency for electron

$\omega \tau = q \mu B$

Drift velocity in cartesian coordinates

$$u_x = \frac{\mu \left|\vec{E}\right|}{1+\omega^2\tau^2}\left[\hat{E}_x + \omega\tau\left(\hat{E}_y\hat{B}_z - \hat{E}_z\hat{B}_y\right) + \omega^2\tau^2\left(\hat{E}\cdot\hat{B}\right)\hat{B}_x\right]$$

$$u_y = \frac{\mu \left|\vec{E}\right|}{1+\omega^2\tau^2}\left[\hat{E}_y + \omega\tau\left(\hat{E}_z\hat{B}_x - \hat{E}_x\hat{B}_z\right) + \omega^2\tau^2\left(\hat{E}\cdot\hat{B}\right)\hat{B}_y\right]$$

$$u_z = \frac{\mu \left|\vec{E}\right|}{1+\omega^2\tau^2}\left[\hat{E}_z + \omega\tau\left(\hat{E}_x\hat{B}_y - \hat{E}_y\hat{B}_x\right) + \omega^2\tau^2\left(\hat{E}\cdot\hat{B}\right)\hat{B}_z\right]$$

We can compute the path integral of the drifting electron

$$\delta_x = \int u_x\,dt = \int \frac{u_x}{u_z}\frac{dz}{dt}\,dt = \int \frac{u_x}{u_z}\,dz$$

$$\delta_y = \int \frac{u_y}{u_z}\,dz \qquad \delta_z = \int \frac{u_z}{u_0}\,dz$$

TPC case: Ez >> Ex,Ey Bz >> Bx,By

$$u_x = \frac{\mu\left|\vec{E}\right|}{1+\omega^2\tau^2}\left[\hat{E}_x + \omega\tau\left(\hat{E}_y\hat{B}_z - \hat{E}_z\hat{B}_y\right) + \omega^2\tau^2\left(\hat{E}\cdot\hat{B}\right)\hat{B}_x\right]$$

$$u_y = \frac{\mu\left|\vec{E}\right|}{1+\omega^2\tau^2}\left[\hat{E}_y + \omega\tau\left(\hat{E}_z\hat{B}_x - \hat{E}_x\hat{B}_z\right) + \omega^2\tau^2\left(\hat{E}\cdot\hat{B}\right)\hat{B}_y\right]$$

$$u_z = \frac{\mu\left|\vec{E}\right|}{1+\omega^2\tau^2}\left[\hat{E}_z + \omega\tau\left(\hat{E}_x\hat{B}_y - \hat{E}_y\hat{B}_x\right) + \omega^2\tau^2\left(\hat{E}\cdot\hat{B}\right)\hat{B}_z\right]$$

Second order expansion:

$$\hat{E}_x \approx \frac{\hat{E}_x}{E_z}$$

$$\hat{E}_z \approx 1 - \frac{1}{2}\hat{E}_x^2 - \frac{1}{2}\hat{E}_y^2$$

$$\frac{u_x}{u_z} = \frac{1}{1+\omega^2\tau^2}\frac{E_x}{E_z} + \frac{\omega\tau}{1+\omega^2\tau^2}\frac{E_y}{E_z} - \frac{\omega\tau}{1+\omega^2\tau^2}\frac{B_y}{B_z} + \frac{\omega^2\tau^2}{1+\omega^2\tau^2}\frac{B_x}{B_z}$$

$$\frac{u_y}{u_z} = \frac{1}{1+\omega^2\tau^2}\frac{E_y}{E_z} - \frac{\omega\tau}{1+\omega^2\tau^2}\frac{E_x}{E_z} + \frac{\omega\tau}{1+\omega^2\tau^2}\frac{B_x}{B_z} + \frac{\omega^2\tau^2}{1+\omega^2\tau^2}\frac{B_y}{B_z}$$